

# A New Figure of Merit and General Properties of Wireless Power Transmission Systems

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**Abstract**—Based on a generalized transmission system this work proposes a new figure of merit to overcome the inaccuracies of previous definitions. Based on the definition of the transmission quality factor we introduce the transmission bandwidth which will be very useful in the analysis and design of load matching networks optimized for maximum power transfer efficiency.

**Index Terms**—Wireless power transmission, Efficiency, Figure of Merit,

## I. INTRODUCTION

The exciting field of Wireless Power Transmission attracted industrial and academic research and, therefore, we see a still increasing number of valuable contributions to this technology. The majority of the research and development work focused on inductive power transmission systems and most of them aim to improve the controllability [1] increase efficiency [2] and spatial freedom.

In section II we describe general properties of transmission systems which are independent of the physics of the transmission path and can, therefore, be applied to the analysis of inductive and capacitive power transmission systems. Based on that we derive in section III an optimum load impedance which maximizes the power transfer efficiency of the system. In section IV we propose a new figure of merit which overcomes the difficulties of the definition of the figure of merit as defined in [3]–[5]. Despite of their broad adoption they are not sufficiently accurate in some relevant practical cases, especially when applied to capacitive power transmission systems. In contrast, the proposed transmission quality factor leads to exact results in all cases. Furthermore we will introduce the transmission bandwidth in section V, a new figure of merit which gains new insights in power transmission systems and supports the analysis and design of load matching networks as will be discussed in section VI.

## II. GENERALIZED POWER TRANSMISSION SYSTEM

In general a transmission system can be considered as an electrical network that links a source and a sink in order to transmit a signal. This signal contains information and power. Without the loss of generality we focus in this work on the transmission of power. A generalized power transmission system is shown in Fig. 1. Assuming that the system is passive, linear and time-invariant we can use 2-port theory and impedance parameters to model the system. We have

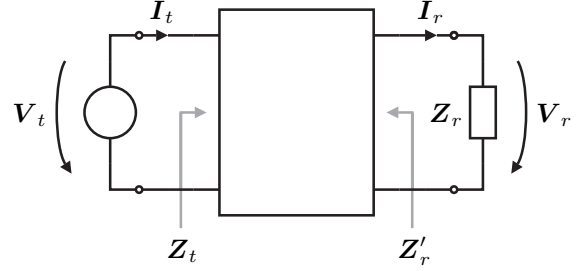


Figure 1. Generalized power transmission system

$$\begin{bmatrix} V_t \\ V_r \end{bmatrix} = \begin{bmatrix} Z_{tt} & -Z_{tr} \\ Z_{rt} & -Z_{rr} \end{bmatrix} \begin{bmatrix} I_t \\ I_r \end{bmatrix} = Z_{tr} \begin{bmatrix} I_t \\ I_r \end{bmatrix} \quad (1)$$

Using (1) the equivalent circuit of the power transmission system shown in Fig.2 can be derived.

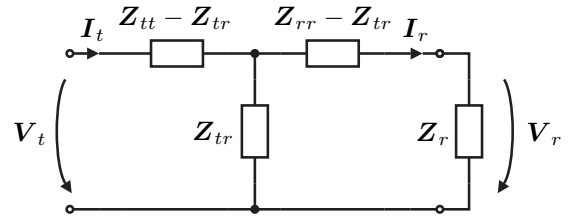


Figure 2. Impedance model of the loaded power transmission system

## III. OPTIMUM EFFICIENCY

In this section we derive the optimum load impedance which maximizes the power transfer efficiency of a general power transmission system. With the previously introduced quantities we can show that the efficiency of the system in Fig. 2 is given by

$$\eta = |H_i|^2 \frac{R_r}{R_t} \quad (2)$$

where

$$H_i = \frac{I_r}{I_t} = \frac{Z_{tr}}{Z_r + Z_{rr}} \quad (3)$$

and

$$Z_t = \frac{Z_{tt}Z_r + \det Z_{tr}}{Z_r + Z_{rr}}. \quad (4)$$

Note that the efficiency depends only on the current gain  $H_i$ , the real part of the load impedance  $R_r$  and the real part of the input impedance  $R_t$ . It depends not on the source impedance of a non-ideal power source. For a

given power transmission system, efficiency optimization is, therefore, only possible by choosing an optimum load impedance.

By representing the impedances by their corresponding real and imaginary parts,  $Z_i = R_i + jX_i$  and  $i \in \{tt, rr, tr\}$ , the efficiency can be written as

$$\eta = \frac{R_r(\alpha - \beta)}{(X_r - \gamma)^2 + (R_r + \alpha)(R_r + \beta)} \quad (5)$$

where

$$\alpha = R_{rr} + \frac{X_{tr}^2}{R_{tt}}, \quad (6)$$

$$\beta = R_{rr} - \frac{R_{tr}^2}{R_{tt}} \quad (7)$$

and

$$\gamma = \frac{X_{tr}R_{tr}}{R_{tt}} - X_{rr}. \quad (8)$$

By solving

$$\frac{\partial}{\partial R_r} \eta = 0 \quad (9)$$

and

$$\frac{\partial}{\partial X_r} \eta = 0 \quad (10)$$

we obtain the optimum load impedance

$$Z_{r,\eta} = R_{r,\eta} + jX_{r,\eta} \quad (11)$$

where

$$R_{r,\eta} = \sqrt{\alpha\beta} = \sqrt{\left(R_{rr} - \frac{R_{tr}^2}{R_{tt}}\right) \left(R_{rr} + \frac{X_{tr}^2}{R_{tt}}\right)} \quad (12)$$

and

$$X_{r,\eta} = \gamma = \frac{X_{tr}R_{tr}}{R_{tt}} - X_{rr}. \quad (13)$$

In [6] it was analytically proved that the extremal (11) is indeed a local maximum of the efficiency (5).

In the special case where the power transmission system is loaded with the optimum load reactance  $X_{r,\eta}$  the expression for the efficiency simplifies to

$$\eta = \frac{\alpha}{\alpha + R_r} - \frac{\beta}{\beta + R_r} \quad (14)$$

The latter equation clearly shows that the efficiency goes to zero for very small as well as for very large values of load resistances  $R_r$ .

#### IV. TRANSMISSION QUALITY FACTOR

As discussed above the figure of merit as defined in [4] has some difficulties which in fact limit its applicability to inductive power transmission systems with negligible  $R_{tr}$  [7]. To overcome these limitations we propose another figure of merit which is applicable to any power transmission system as follows

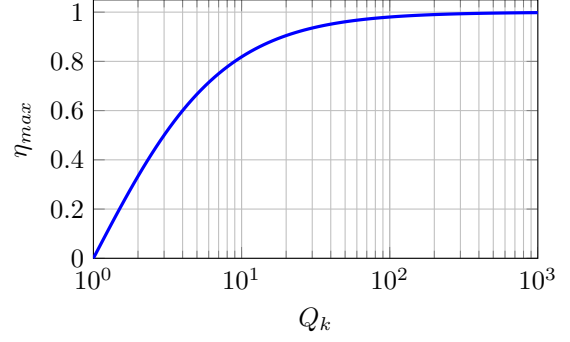


Figure 3. Efficiency as a function of the power transmission quality factor  $Q_k$

$$Q_k = \sqrt{\frac{\alpha}{\beta}} = \sqrt{\frac{1 + \frac{X_{tr}^2}{R_{tt}R_{rr}}}{1 - \frac{R_{tr}^2}{R_{tt}R_{rr}}}}. \quad (15)$$

With this definition of  $Q_k$  the maximum efficiency of an power transmission system can be written as

$$\eta_{max} = \frac{Q_k - 1}{Q_k + 1}. \quad (16)$$

The efficiency as a function of the power transmission quality factor  $Q_k$  is plotted in Fig. 3. Using (16) the maximum achievable power transmission efficiency from any transmission system can be determined easily based on the known impedance parameters of the system.

Another definition of a figure of merit ( $FOM$ ) can be found in [3]–[5], for example. Using the nomenclature of this work the  $FOM$  can be written as

$$FOM = \sqrt{\frac{X_{tr}^2}{R_{tt}R_{rr}}}. \quad (17)$$

Despite of its broad adoption one should carefully check its validity before using it for the analysis of a particular wireless power transmission system. In fact, the definition (17) does not consider a resistive component of the shunt impedance  $Z_{tr}$ . The reason for this lies in the derivation of the  $FOM$  which typically relies on a transformer equivalent circuit to model the electrical behavior of magnetically coupled coils in an inductive power transmission system. In the transformer equivalent circuits used for the derivation of the  $FOM$  resistive losses were usually modeled as series resistors while the shunt impedance is considered to be purely reactive which is in most cases a simplification whose validity has to be checked. In [7] simulated and measured values for  $R_{tr}$  in an inductive power transmission system have been presented. Although approximately one order of magnitude smaller than the series resistances  $R_{tt}$  and  $R_{rr}$ ,  $R_{tr}$  had to be considered in the model to be in good accordance with the measured and simulated results. While excluding  $R_{tr}$  in the circuit model of inductive systems might lead to only small errors the opposite is the case in capacitive systems. The

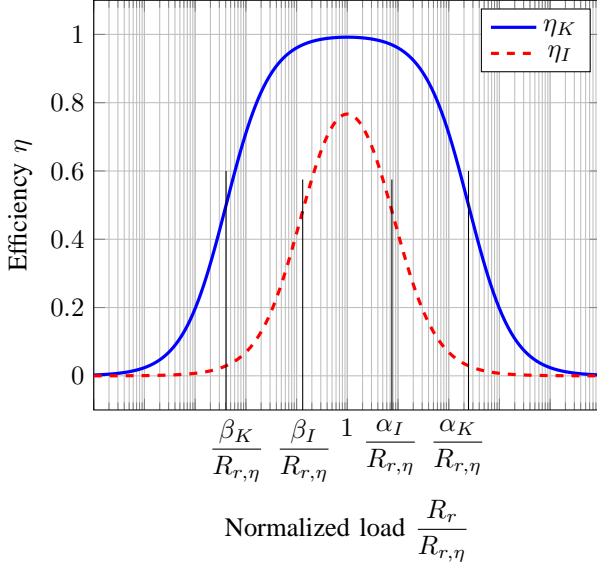


Figure 4. Efficiency over  $R_r$  for different values of  $Q_k$  and  $X_r = X_{r,\eta}$

examples given in [6] clearly show, that  $R_{tr}$  in capacitive systems are in the same order of magnitude than the series resistances and can, therefore, not be neglected in the circuit model and analysis. In contrast, the proposed transmission quality factor (15) can be applied in any cases and leads to exact results. Consequently,  $Q_k$  should be used instead of  $FOM$ . At the end of this section we want to point out, that for  $R_{tr} = 0$  a simple relation between  $Q_k$  and  $FOM$  exists:

$$Q_k = \sqrt{1 + FOM^2} \quad (18)$$

The latter equation can be used to transform previous results obtained by other authors to the context of this work.

## V. TRANSMISSION BANDWIDTH

In the previous section we found the optimum load resistance with respect to efficiency. However, in a practical application power has to be delivered to load which may vary over a wide range. Therefore we now investigate, how the efficiency changes when the load deviates from the optimum. From here we assume that the system is loaded by the optimum load reactance  $X_{r,\eta}$ . Fig. 4 shows two efficiency curves for two different values of  $Q_k$  together with the corresponding values of  $\alpha$  and  $\beta$  normalized to the optimum load resistance  $R_{r,\eta}$  on a semi-log scale. The curves are symmetrical and the efficiency at  $\alpha/R_{r,\eta}$  equals the efficiency at  $\beta/R_{r,\eta}$ . Hence we have

$$\log\left(\frac{\alpha}{R_{r,\eta}}\right) = -\log\left(\frac{\beta}{R_{r,\eta}}\right) \quad (19)$$

and

$$\eta_\alpha = \eta_\beta = \frac{1}{2} \frac{Q_k^2 - 1}{Q_k^2 + 1}. \quad (20)$$

We can now define the difference

$$\frac{\alpha}{R_{r,\eta}} - \frac{\beta}{R_{r,\eta}} \quad (21)$$

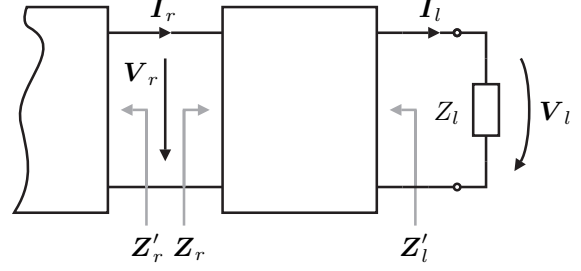


Figure 5. Power transmission system with load matching network

as the transmission bandwidth  $B_k$ . Using the definitions  $R_{r,\eta} = \sqrt{\alpha\beta}$  and  $Q_k = \sqrt{\alpha/\beta}$  from above we find

$$B_k = Q_k - \frac{1}{Q_k}. \quad (22)$$

For high values of  $Q_k$  the efficiency at the band limits ( $\alpha/R_{r,\eta}$ ,  $\beta/R_{r,\eta}$ ) is approximately 0.5 and drops further for low values of  $Q_k$ .

The limits of the load resistance range where the efficiency of a power transmission system with given  $\alpha$  and  $\beta$  exceeds a specified minimum,  $\eta \geq \eta_g$ , can be found by evaluation of

$$R_{r,1,2} = \frac{1}{2\eta_g} [\alpha(1 - \eta_g) - \beta(1 + \eta_g) \pm \sqrt{(\beta - \alpha) [\beta(1 + \eta_g)^2 - \alpha(1 - \eta_g)^2]}] \quad (23)$$

## VI. INFLUENCE OF A LOAD MATCHING NETWORK

In most practical cases a load matching network is connected to the power transmission system as shown in Fig. 5 in order to transform the actual load resistance to the optimum value. The matching network can also be described using impedance parameters

$$Z_A = \begin{bmatrix} jx_{1,1} & -jx_{1,2} \\ jx_{1,2} & -jx_{2,2} \end{bmatrix}. \quad (24)$$

The cascade connection of transmission system and a loss-free matching network was analyzed in [6]. It can be shown that

$$\alpha_{tl} = \delta\alpha \quad (25)$$

and

$$\beta_{tl} = \delta\beta \quad (26)$$

where  $\alpha_{tl}$  and  $\beta_{tl}$  correspond to (6) and (7) for the cascade connection and the factor  $\delta$  is given by

$$\delta = \frac{x_{1,2}^2}{x_{1,1}^2 - X_{rr}^2 - 2\gamma(X_{rr} + x_{1,1}) + R_{rr}(\alpha + \beta) - R_{rr}^2}. \quad (27)$$

Using (25), (26) and (27) we can determine the transmission quality factor

$$Q_{k_{tl}} = \sqrt{\frac{\alpha_{tl}}{\beta_{tl}}} = \sqrt{\frac{\delta\alpha}{\delta\beta}} = \sqrt{\frac{\alpha}{\beta}} = Q_k. \quad (28)$$

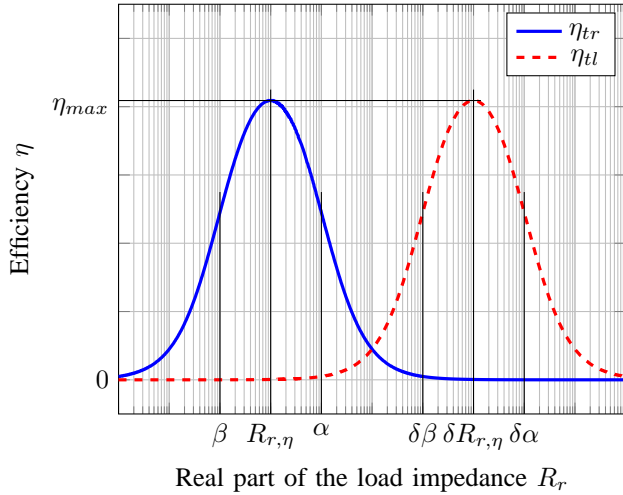


Figure 6. Shifting of the efficiency curve along the R-axis when using a loss-free matching network terminated by the optimum load reactance

as well as the optimum load

$$R_{l,\eta} = \sqrt{\alpha_{tl}\beta_{tl}} = \sqrt{\delta\alpha\delta\beta} = \delta R_{r,\eta}. \quad (29)$$

Furthermore, if we assume that the cascaded system is loaded with the optimum load reactance  $X_{l,\eta}$  we get for the maximum efficiency

$$\begin{aligned} \eta_{tl} &= \frac{\alpha_{tl}}{\alpha_{tl} + R_r} - \frac{\beta_{tl}}{\beta_{tl} + R_r} \\ &= \frac{\alpha}{\alpha + \frac{R_r}{\delta}} - \frac{\beta}{\beta + \frac{R_r}{\delta}}. \end{aligned} \quad (30)$$

This analysis makes clear, that the transmission quality factor and the transmission bandwidth do not change if a loss-free matching network which terminates into  $Z_{l,\eta}$  is inserted. That means that the shape of the efficiency curve remains unchanged and the curve is simply shifted along the resistance axis.

## VII. CONCLUSIONS

In this paper we have introduced the transmission quality factor which is more general and accurate than the figure of merit used before. Based on that we defined the transmission bandwidth as another new property of power transmission systems which allows to determine easily the load range within the system can be efficiently operated. Examples for inductive and capacitive power transmission systems can be found in [6]. In a subsequent paper we will discuss how the transformation invariance of the transmission quality factor and bandwidth introduced in this paper can be used advantageously in the design of matching networks including two and multiple coil setups.

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